$$(5) \quad \xi_{m+1} < M q_m^{n-2}.$$

Hence the inequality, as given by Borel,

$$(6) \quad a_{m+1} < M q_m^{n-2}.$$

4. When $n=1$, ξ is rational and inequality (6) takes the form

$$a_{m+1} < M/q_m.$$

Since q_m increases without limit with m, by taking m great enough a_{m+1} may be made less than any assigned positive number, however small. Now this involves a contradiction since always $a_{m+1} \geqq 1$. Hence the elementary

THEOREM. *A rational number is represented by a terminating continued fraction.*

5. When $n=2$, the inequality (6) becomes

$$(7) \quad a_{m+1} < M.$$

To determine M explicitly let

$$f(x) \equiv ax^2 + bx + c, \quad b^2 - 4ac \equiv D > 0,$$

and let ξ represent either of the values $(-b \pm \sqrt{D})/a$. Since

$$\left| \frac{p_m}{q_m} - \xi \right| < 1, \qquad m = 0, 1, 2, \ldots.$$

two numbers α, β can be chosen in the interval $\xi-1$, $\xi+1$ so that ξ and every convergent p_m/q_m, $m=0, 1, 2, \ldots$, shall lie in the interval α, β. Hence when for M the greater of the values $|f'(\xi-1)|$, $|f'(\xi+1)|$ is taken, then in the interval α, β, $|f(x)| < M$. We find that

$$M = 2(|a| + \sqrt{D}).$$

Comparing this with inequality (7) we have the following

THEOREM. *In the continued fraction representing either quadratic irrationality $(-b \pm \sqrt{D})/a$, a, b, D integers, $D>0$, every partial denominator after the first is less than $2(|a| + \sqrt{D})$.*

6. Denote by ϵ a small positive number. Then α and β may be chosen in the interval $\xi-\epsilon$, $\xi+\epsilon$ so that for i great enough ξ and every convergent of order greater than i lies in the interval α, β. As in the preceding instance take for M the greater of the two numbers $|f'(\xi-\epsilon)|$, $|f'(\xi+\epsilon)|$. Then in the interval α, β,

$$|f'(x)| < M, \quad M = 2(|a|\epsilon + \sqrt{D}).$$

Now let $2\sqrt{D}+d$ be the integer next greater than $2\sqrt{D}$ and choose ε so that

$$\varepsilon<\frac{d}{2\mid a\mid}, \text{ whence } M<2\sqrt{D}+d.$$

Hence by inequality (7), since a_{m+1} is an integer,

$$a_{m+1}<2\sqrt{D}$$

for every m, $m>i$. This is the well known

THEOREM. *In the continued fraction representing either quadratic irrationality* $(-b\pm\sqrt{D})/a$, a, b, D *integers,* $D>0$, *every partial denominator from one of a certain rank on, is less than* $2\sqrt{D}$.

7. The more general theorem of which this is a special case is the following

THEOREM. *In the continued fraction representing the real number* ξ, *where* ξ *is the root of an irreducible equation with integral coefficients* $f(x)=0$ *of degree* n, *every partial denominator* a_{m+1}, *from one of a certain rank on, satisfies the inequality* $a_{m+1}<\mid f'(\xi)\mid q_m^{n-2}$.

PROPERTIES OF THE FUNCTION $(1+a)^x$.

By ANTONIO LLANO. Scranton, Pa.

The following demonstrations of some well known theorems are submitted as being simpler and more systematic than those usually given. The binomial $1+a$ is supposed positive, or $a>-1$.

THEOREM I. *If* $x>1$, *then* $(1+a)^x>1+ax$.

Let $x=u/v$, where $u>v$, and put $1+a=z^v$. We have

$$\frac{z^u-1}{z^v-1}=\frac{z^{u-1}+z^{u-2}+\ldots+1}{z^{v-1}+z^{v-2}+\ldots+1}=1+\frac{z^v+z^{v+1}+\ldots+z^{u-1}}{z^{v-1}+z^{v-2}+\ldots+1}\ldots(1).$$

If $z>1$, the fraction in the final member is less than $\frac{(u-v)z^v}{vz^{v-1}}$.

$$\therefore \frac{z^u-1}{z^v-1}>1+\frac{(u-v)z}{v}>1+\frac{u-v}{v}, \text{ namely, } \frac{u}{v}\ldots(2).$$

$$\therefore z^u>1+(z^v-1)\frac{u}{v}; \text{ or, } (1+a)^{u/v}>1+a\frac{u}{v}.$$

If $z<1$, the final member of (1) is less than $1+(u-v)z/v$, and the character of the inequalities in (2) is reversed; but, as z^v-1 negative, the inequalities

are again reversed on multiplying by z^v-1, and the result is the same as before.

THEOREM II. *If x lies between 0 and 1, then* $(1+a)^x<1+ax$.

Let $x=\frac{v}{u}$, $v<u$, and put $(1+a)^{v/u}=1+y$. Then

$$1+a=(1+y)^{u/v}>1+\frac{u}{v}y. \quad \text{(Theorem I).}$$

$$\therefore y<a\frac{v}{u}\text{; and, therefore, } (1+a)^{v/u}<1+a\frac{v}{u}.$$

THEOREM III. *If x is any positive quantity, then* $(1+a)^{-x}>1-ax$.

First. Let x be greater than 1, and equal to u/v. Put $(1+a)^{-x}=1+y$. Then,

$$(1+a)^{-x}=\left(\frac{1}{1+a}\right)^{u/v}=\left(1-\frac{a}{1+a}\right)^{u/v}>1-\frac{a}{1+a}\frac{u}{v} \quad \text{(Theorem I).}$$

$$\therefore 1+y>1-\frac{a}{1+a}\frac{u}{v}\text{; } y>-\frac{a}{1+a}\frac{u}{v}>-a\frac{u}{v}\text{;}$$

$$\text{and } (1+a)^{-x}>1-a\frac{u}{v}, \text{ or } >1-ax.$$

(If $a>0$, then $\frac{a}{1+a}<a$, and $-\frac{a}{1+a}>a$. If $a<0$ and equal to $-b$, then, since $1+a$ is supposed >0, $-\frac{a}{1+a}=\frac{b}{1-b}>b$, or $>-a$).

Second. Let x be less than 1, and equal to v/u. Then

$$(1+a)=(1+y)^{-(u/v)}>1-y\frac{u}{v} \text{ (by preceding case).}$$

Whence, as before,

$$-y\frac{u}{v}<a,\ y>-a\frac{v}{u}, \text{ or } >-ax.$$

THEOREM IV. *For all values of x greater than 1 or less than 0,* $(1+a)^x<1+ax(1+a)^{x-1}$; *and for all values of x between 0 and 1,* $(1+a)^x>1+ax(1+a)^{x-1}$.

If $x>1$, or <0, we have (Theorems I and III),

$$\left(\frac{1}{1+a}\right)^x=\left(1-\frac{a}{1+a}\right)^x>1-ax\frac{1}{1+a}.$$

$$\therefore 1>(1+a)^x-ax(1+a)^{x-1},\ (1+a)^x<1+ax(1+a)^{x-1}.$$

The second part of the theorem is similarly proved, by the aid of Theorem II.*

THEOREM V. *For positive values of x, the functions* $\left(1+\frac{1}{x}\right)^x$ *and* $\left(1-\frac{1}{x}\right)^x$ *are both increasing functions.*

* The demonstration of the theorem just proved the same in principle as that given in Chrystal's *Textbook of Algebra*, and is given here merely for the sake of completeness.

Let x_1 and x_2 be two values of x; assume $x_1 > x_2$, and put $\left(1+\frac{1}{x_1}\right)^{x_1} \div \left(1+\frac{1}{x_2}\right)^{x_2} = \lambda$. Then,

$$\left(1+\frac{1}{x_1}\right)^{x_1} = \lambda\left(1+\frac{1}{x_2}\right)^{x_2},$$

whence

$$\lambda^{1/x_2}\left(1+\frac{1}{x_2}\right) = \left(1+\frac{1}{x_1}\right)^{x_1/x_2} > 1+\frac{1}{x_2} \text{ (Theorem I)};$$

and, therefore,

$$\lambda > 1, \text{ or } \left(1+\frac{1}{x_1}\right)^{x_1} > \left(1+\frac{1}{x_2}\right)^{x_2}.$$

Similarly for $\left(1-\frac{1}{x}\right)^{x}$.

It follows that $\left(1-\frac{1}{x}\right)^{-x}$ increases as x increases.

THEOREM VI. *The functions* $\left(1+\frac{1}{x}\right)^{x}$ *and* $\left(1-\frac{1}{x}\right)^{x}$ *have each a limit as* x *increases indefinitely, and these limits are the reciprocal of each other.*

For we have,

$$\left(1+\frac{1}{x}\right)^{x}\left(1-\frac{1}{x}\right)^{x} = \left(1-\frac{1}{x^2}\right)^{x} < 1, \text{ but } > 1-\frac{1}{x} \text{ (Theorem I)}.$$

This shows that neither function can increase indefinitely, since they are both increasing functions (Theorem IV).

From the inequalities

$$1 > \left[1+\frac{1}{x}\right]^{x}\left[1-\frac{1}{x}\right]^{x} > 1-\frac{1}{x},$$

follows

$$\lim\left[1+\frac{1}{x}\right]^{x} \lim\left[1-\frac{1}{x}\right]^{x} = 1.$$

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

181. Proposed by J. F. LAWRENCE, Breckenridge, Mo.

Show that $\phi(1)\frac{x}{1+x^2}-\phi(3)\frac{x^3}{1+x^6}+\phi(5)\frac{x^5}{1+x^{10}}-\text{ad inf.}=\frac{x(1-x^2)}{(1+x^2)^2}$, $\phi(n)$ being the number of integers less than n and prime to it. [From Hall and Knight's *Higher Algebra*, page 358].

Solution* by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

$$\phi(1)\frac{y}{1-y^2}+\phi(3)\frac{y^3}{1-y^6}+\phi(5)\frac{y^5}{1-y^{10}}+\cdots\cdots$$

$$=\frac{y}{1-y^2}+\frac{2y^3}{1-y^6}+\frac{4y^5}{1-y^{10}}+\frac{6y^7}{1-y^{14}}+\frac{8y^9}{1-y^{18}}+\cdots$$

$$=y(1+y^2+y^4+y^6+y^8+\cdots)+2y^3(1+y^6+\cdots)+4y^5(1+y^{10}+\cdots)$$

$$+6y^7(1+y^{14}+\cdots)+8y^9(1+y^{18}+\cdots)+\cdots=y+3y^3+5y^5+7y^7+9y^9+\cdots$$

$$=y(1+y^2)(1+2y^2+3y^4+4y^5+5y^8+\cdots)=y(1+y^2)(1-y^2)^{-2}=\frac{y(1+y^2)}{(1-y^2)^2}.$$

Let $y=x\sqrt{(-1)}$.

$$\therefore \phi(1)\frac{x}{1+x^2}-\phi(3)\frac{x^3}{1+x^6}+\phi(5)\frac{x^5}{1+x^{10}}-\phi(7)\frac{x^7}{1+x^{14}}+\cdots=\frac{x(1-x^2)}{(1+x^2)^2}.$$

182. Proposed by J. F. LAWRENCE, Breckenridge, Mo.

Find the values of $x_1, x_2, x_3, \ldots, x_n$ which satisfy the following system of simultaneous equations:

$$\frac{x_1}{a_i-b_1}+\frac{x_2}{a_i-b_2}+\cdots+\frac{x_n}{a_i-b_n}=1 \quad (i=1, 2, \ldots, n).$$

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

The equation $\frac{x_1}{\theta-b_1}+\frac{x_2}{\theta-b_2}+\cdots+\frac{x_n}{\theta-b_n}=1-\frac{(\theta-a_1)(\theta-a_2)\cdots(\theta-a_n)}{(\theta-b_1)(\theta-b_2)\cdots(\theta-b_n)}$,

is an equation of the $(n-1)$th degree in θ and is satisfied by the n values $a_1, a_2, \ldots, a_n$. Multiply each side by $\theta-b_1$, and then place $\theta=b_1$.

*The general term should be considered. For $8y^9$ read $6y^9$. Editor D.

$$\therefore x_1 = -\frac{(b_1-a_1)(b_1-a_2)\ldots\ldots(b_1-a_n)}{(b_1-b_2)(b_1-b_3)\ldots\ldots(b_1-b_n)},\quad x_r = -\frac{(b_r-a_1)(b_r-a_2)\ldots\ldots(b_r-a_n)}{(b_r-b_1)(b_r-b_2)\ldots\ldots(b_r-b_n)}.$$

GEOMETRY.

203. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Show that two parabolae can always be drawn through the vertices of a triangle to touch its circumcircle at a vertex, and that the axes of these pairs of curves are orthogonal. Show that any triangle may be circumscribed by a conic so that the tangents at each vertex are parallel to the opposite side.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

In order that $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$ may be a parabola we must have

$$l^2a^2 + m^2b^2 + n^2c^2 = 2ablm + 2bcmn + 2acln.$$

For tangency at a vertex we must have $l\beta + m\alpha = 0$, also $\alpha\sin B + \beta\sin A = 0$.

$\therefore l/m = \sin A/\sin B = a/b$, or $l = ma/b$.

$\therefore m^2a^2/b^2 + m^2b^2 + n^2c^2 = 2a^2m^2 + 2bcmn + 2a^2cmn/b$.

$\therefore m^2(a^2-b^2)^2 + n^2b^2c^2 = (2b^3c + 2a^2bc)mn$.

$$\therefore n = \frac{(a\pm b)^2}{bc}m.$$

$\therefore ac\beta\gamma + bc\gamma\alpha + (a\pm b)^2\alpha\beta = 0$ is the equation to the two parabolae.

If two of the sides be taken as axes, the Cartesian equation is $(x\pm y)^2 + bx + ay = 0$. In order that the tangents at each vertex may be parallel to the opposite sides we must have $l/m = b/a$, $m/n = c/b$.

$\therefore \beta\gamma/a + \gamma\alpha/b + \alpha\beta/c = 0$ is the conic required, and this conic is an ellipse.

II. Solution by G. W. GREENWOOD, B. A., Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

Let the triangle be formed by the lines α, β, γ. The tangent at the vertex $\alpha\beta$ to the circumcircle is

$$a\beta + b\alpha = 0.$$

The equation of any circumconic having this line for a tangent may be written

$$a\beta\gamma + b\gamma\alpha + \lambda\alpha\beta = 0.$$

The coördinates of the center, *i. e.*, of the pole of the line at infinity, are found to be

$$\lambda ac - a^3 + ab^2,\quad \lambda bc - b^3 + a^2b,\quad \lambda a^2 + \lambda b^2 - \lambda^2 c.$$

If this point lies on the line at infinity, the conic will be a parabola. This condition gives $\lambda = (a\pm b)^2/c$, and hence there are two parabolae satisfying this condition. The coördinates of the centers become

$$2ab(b\pm a),\quad 2ab(a\pm b),\quad \mp 2ab(a\pm b)^2/c.$$

From the first two coördinates we see that the axes of these parabolae are the bisectors of the angle at $\alpha\beta$.

Problem 204 was also solved by D. B. Northrup, Schenectady, N. Y.

205. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Having given any two systems of conjugate semi-diameters of an ellipsoid, the parallelopiped which has any three for continuous edges is equal to that which has the other three for continuous edges.

Remark by H. B. LEONARD, A.B., Chicago, Ill.

Solved in C. Smith's "An Elementary Treatise on Solid Geometry," page 76.

CALCULUS.

166. Proposed by T. N. HAUN, Mohawk, Tenn.

Find the volume of the solid formed by the revolution of the curve $(y^2+x^2)=a^2(x^2-y^2)$ round the axis of x.

II. Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

This equation represents singly the two lines $y=\pm x\sqrt{(a^2-1)/(a^2+1)}$, and the surface generated is simply a right circular cone whose area, volume, bounded by the plane $x=c$, is simply $\frac{1}{3}\pi c^3(a^2-1)/(a^2+1)$, and where $a\neq 1$.

168. Proposed F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

The tangent of what Cartesian curve makes an x-intercept always m times as long as as the corresponding y-intercept?

II. Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

Let the tangent at $P(x, y)$ meet the axes in the points A and B. Let the perpendicular from P to OY meet it at M. Call θ the angle APM.

Then $PB=m.PA$, $OM=m.AM$; *i. e.*, $y=m.PM.\tan\theta=mx(dy/dx)$.

$\therefore y^m=cx$.

AVERAGE AND PROBABILITY.

145. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

In each quadrant of a given circle, a circle is described at random. A point is taken at random in each of these circles. What is the average area of the quadrilateral formed by joining with straight lines these four points?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

Let P, Q, R, S be the centers of the four circles. The average area of the required quadrilateral is the same as the average area of the quadrilateral formed by joining the centroids of the four circles, or their centers.

Let $OP=s$, $PC=t$, $OQ=u$, $QF=v$, $OR=w$, $RE=x$, $OS=y$, $DS=z$, $OG=a$, $\angle AOP=\theta$, $\angle AOQ=\phi$, $\angle BOR=\psi$, $\angle BOS=\rho$.

The limits of t, v, x, and z are 0 and $a(\sqrt{2}-1)=m$; of s, $t\sqrt{2}$ and $a-t$; of u, $v\sqrt{2}$ and $a-v$; of w, $x\sqrt{2}$ and $a-x$; of y, $z\sqrt{2}$ and $a-z$; of θ, $\sin^{-1}(t/s)=\theta''$ and $\frac{1}{2}\pi-\theta''=\theta'$; of ϕ, $\sin^{-1}(v/u)=\phi''$ and $\frac{1}{2}\pi-\phi''=\phi'$; of ψ, $\sin^{-1}(x/w)=\psi''$ and $\frac{1}{2}\pi-\psi''=\psi'$; of ρ, $\sin^{-1}(z/y)=\rho''$ and $\frac{1}{2}\pi-\rho''=\rho'$.

The area of $PQRS=\frac{1}{2}[su\sin(\theta+\phi)+uw\sin(\phi+\psi)+wy\sin(\psi+\rho)+sy\sin(\theta+\rho)]=A$.

$$\therefore \triangle=\frac{\int_0^m\int_0^m\int_0^m\int_0^m\int_{t\sqrt{2}}^{a-t}\int_{v\sqrt{2}}^{a-v}\int_{x\sqrt{2}}^{a-x}\int_{z\sqrt{2}}^{a-z}\int_{\theta'}^{\theta'}\int_{\phi'}^{\phi'}\int_{\psi'}^{\psi'}\int_{\rho'}^{\rho'} A\,I}{\int_0^m\int_0^m\int_0^m\int_0^m\int_{t\sqrt{2}}^{a-t}\int_{v\sqrt{2}}^{a-v}\int_{x\sqrt{1}}^{a-x}\int_{z\sqrt{2}}^{a-z}\int_{\theta'}^{\theta'}\int_{\phi'}^{\phi'}\int_{\psi'}^{\psi'}\int_{\rho'}^{\rho'} I}$$

where $I=dt\,dv\,dx\,dz\,sds\,udu\,wdw\,ydy\,d\theta\,d\phi\,d\psi\,d\rho$.

$$\triangle=\frac{6^8}{(3\pi-28+16\sqrt{2})^4a^{12}}$$

$$\times\int_0^m\int_0^m\int_0^m\int_0^m\int_{t\sqrt{2}}^{a-t}\int_{v\sqrt{2}}^{a-v}\int_{x\sqrt{2}}^{a-x}\int_{z\sqrt{2}}^{a-z}\int_{\theta'}^{\theta'}\int_{\phi'}^{\phi'}\int_{\psi'}^{\psi'}\int_{\rho'}^{\rho'} A\,I$$

$$\triangle=\frac{648}{(3\pi-28+16\sqrt{2})^2a^6}\Big[\int_0^m\int_{t\sqrt{2}}^{a-t}\int_{\theta'}^{\theta'}\int_0^m\int_{v\sqrt{2}}^{a-v}\int_{\phi'}^{\phi'}$$

$$\times s^2u^2\sin(\theta+\phi)dt\,ds\,d\theta\,dv\,du\,d\phi$$

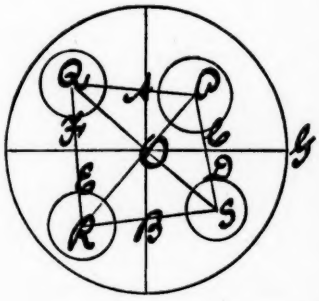
$$+\int_0^m\int_{v\sqrt{2}}^{a-v}\int_{\phi'}^{\phi'}\int_0^m\int_{x\sqrt{2}}^{a-x}\int_{\psi'}^{\psi'}$$

$$\times u^2w^2\sin(\phi+\omega)dv\,du\,d\phi\,dx\,dw\,d\psi$$

$$+\int_0^m\int_{x\sqrt{2}}^{a-x}\int_{\psi'}^{\psi'}\int_0^m\int_{z\sqrt{2}}^{a-z}\int_{\rho'}^{\rho'}$$

$$\times w^2y^2\sin(\psi+\rho)dx\,dw\,d\psi\,dz\,dy\,d\rho$$

$$+\int_0^m\int_{t\sqrt{2}}^{a-t}\int_{\theta'}^{\theta'}\int_0^m\int_{z\sqrt{2}}^{a-z}\int_{\rho'}^{\rho'} s^2y^2\sin(\theta+\rho)dt\,ds\,d\theta\,dz\,dy\,d\rho\Big]$$



$$=\frac{9(3729-2624\sqrt{2})a^2}{25(3\pi-28+16\sqrt{2})^2}=\left(\frac{3}{5}\frac{a}{}\right)^2\left(\frac{32\sqrt{2}-41}{3\pi-28+16\sqrt{2}}\right)^2=\tfrac{9}{25}a^2, \text{ nearly,}$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

187. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Express by radicals the roots of $x^7+px^5+\frac{2}{7}p^2x^3+\frac{1}{49}p^3x+r=0$.

188. Proposed by GUY SCHUYLER.

$xy+ab=2ax$, $x^2y^2+a^2b^2=2b^2y^2$.

GEOMETRY.

209. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Find by a geometrical method the maximum value of $\sin\theta\cos\theta\cos2\theta$.

210. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Let ADC be a triangle with angle $C=120°$, and let the interior bisector of angle C meet AD in B. Prove that $2.CB$ is the harmonic mean between CA and CD.

211. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Prove the validity of the following construction of an inscribed regular pentagon and regular decagon: Draw any two perpendicular radii of the given circle with center C. Call E the end of one radius CE and M the middle point of the perpendicular radius CM. Take the point R on CM produced through C such that $RCM=EM$. Then RC=side of inscribed regular decagon, RE=side of inscribed regular pentagon.

CALCULUS.

172. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

$$\text{Solve } x\frac{dy}{dx}=\frac{y}{y^{-1}-\log x}.$$

DIOPHANTINE ANALYSIS.

119. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

If p be any prime number and n any positive integer, the congruence

$x^{p^n} \equiv x (\mathrm{mod} p^n)$ has p and p solutions $\mathrm{mod} p^n$. Hence the congruence defines the Galois field of order p^n if and only if $n=1$.

AVERAGE AND PROBABILITY.

148. Proposed by M. C. RORTY, Boston, Mass.

Assuming n points to fall at random upon a circle of circumference a, what is the probability of m or more points falling within a length b upon this circumference.

NOTE. This problem has practical application in determining the probability of accidental rushes of telephone calls as distinct from those rushes which are due to commercial causes. The solution for m or more points falling within a specified length b is known. The problem presented above differs from this in that a solution is required for *any* length b.

149. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Three points are taken at random on the convex surface of a right cone. Find the probability that the section of the cone made by the plane passing through them is a complete ellipse.

MISCELLANEOUS.

142. Proposed by R. A. WELLS, Franklin College, New Athens, O.

Find a general expression for the value of θ such that when θ is one of the acute angles of a right triangle, the three sides of the triangle will be commensurable.

NOTES.

Dr. C. N. Haskins has been appointed Instructor in Mathematics at Yale University.

Dr. A. B. Pierce has been appointed Instructor in Mathematics at the University of Michigan.

Dr. G. B. Halsted has been elected Professor of Mathematics at Kenyon College, Gambier, O.

Mr. A. C. Minear has been appointed Professor of Mathematics in the University of Southern California, to succeed Mr. Paul Arnold, resigned.

Dr. H. C. DeMott, Principal of the Preparatory Department of the Illinois Wesleyan University, has been promoted to the University chair of mathematics.

In line 14 of Professor Himel's Note in the October number, read "Thus, in the above example, if 2 be true."

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A GENERALIZATION OF SYMMETRIC AND SKEW-SYMMETRIC DETERMINANTS.

By DR. L. E. DICKSON, The University of Chicago.

1. Examples of symmetric and skew-symmetric determinants are, respectively,

$$Y=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}, \quad K=\begin{vmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{vmatrix}.$$

A direct generalization is afforded by the following determinant:

$$\Delta=\begin{vmatrix} a_{11} & a_{12}+iA_{12} & a_{13}+iA_{13} \\ a_{12}-iA_{12} & a_{22} & a_{23}+iA_{23} \\ a_{13}-iA_{13} & a_{23}-iA_{23} & a_{33} \end{vmatrix}.$$

For determinants like Δ, and D below, with a_{ij} and A_{ij} real, we have the

THEOREM. *A determinant with symmetrically conjugate elements is real.*

For proof we reflect Δ on the main diagonal and obtain an equal determinant whose elements are conjugate with the corresponding ones of the given determinant. From $\Delta=\bar{\Delta}$ follows $\Delta=$real.

2. Expanding Δ, we get $\Delta=Y+Q$, where

$$Q\equiv -a_{11}A_{23}^2-a_{22}A_{13}^2-a_{33}A_{12}^2+2a_{12}A_{13}A_{23}+2a_{23}A_{12}A_{13}-2a_{13}A_{12}A_{23}.$$

Regarding A_{12}, A_{13}, and A_{23} as variables, we seek the conditions under which Q shall factor into linear expressions. To fix the ideas, let $a_{11}\neq 0$. Completing the square in A_{23}, we get finally

$$-a_{11}Q=(a_{11}A_{23}-a_{12}A_{13}+a_{13}A_{12})^2-A^2_{13}(a^2_{12}-a_{11}a_{22})$$
$$-A^2_{12}(a^2_{13}-a_{11}a_{33})-2A_{12}A_{13}(a_{11}a_{23}-a_{12}a_{13}).$$

The last three terms, with their signs changed, form a perfect square if and only if

$$(a^2_{12}-a_{11}a_{22})(a^2_{13}-a_{11}a_{33})-(a_{11}a_{23}-a_{12}a_{13})^2=0,$$

that is, if $a_{11}Y=0$. For $Y=0$ we therefore have

$$-a_{11}Q=(P+T)(P-T),\quad P=a_{11}A_{23}-a_{12}A_{13}+a_{13}A_{12},$$
$$T=A_{13}\sqrt{(a^2_{12}-a_{11}a_{22})}+A_{12}\sqrt{(a^2_{13}-a_{11}a_{33})},$$

the signs of the radicals being suitably chosen.

Interpreting the A_{ij} as homogeneous point-coördinates in the plane, we conclude that the equation $\triangle=Y$ represents a pair of straight lines if and only if $Y=0$.

3. Consider next the symmetrically conjugate determinant

$$D=\begin{vmatrix} a_{11} & a_{12}+iA_{12} & a_{13}+iA_{13} & a_{14}+iA_{14} \\ a_{12}-iA_{12} & a_{22} & a_{23}+iA_{23} & a_{24}+iA_{24} \\ a_{13}-iA_{13} & a_{23}-iA_{23} & a_{33} & a_{24}+iA_{34} \\ a_{14}-iA_{14} & a_{24}-iA_{24} & a_{34}-iA_{34} & a_{44} \end{vmatrix},$$

and the included symmetric and skew-symmetric determinants

$$a=|\,a_{ij}\,|,\quad A=\begin{vmatrix} 0 & A_{12} & A_{13} & A_{14} \\ -A_{12} & 0 & A_{23} & A_{24} \\ -A_{13} & -A_{23} & 0 & A_{34} \\ -A_{14} & -A_{24} & -A_{34} & 0 \end{vmatrix}.$$

By the theory of skew-symmetric determinants,

$$K\equiv 0,\quad A\equiv(A_{12}A_{34}-A_{13}A_{24}+A_{14}A_{23})^2.$$

To expand D, we write $a_{ii}=a_{ii}+0$, thus obtaining each element expressed as an algebraic sum of two terms. Hence D equals the sum of 16 determinants of the fourth order, among which occur a and A. In view ot the theorem of §1, 8 of the determinants cancel, namely, those in which the elements of an *odd* number of columns contain the factor i. Each of the remaining 6 determinants have the factor i^2. Their sum is